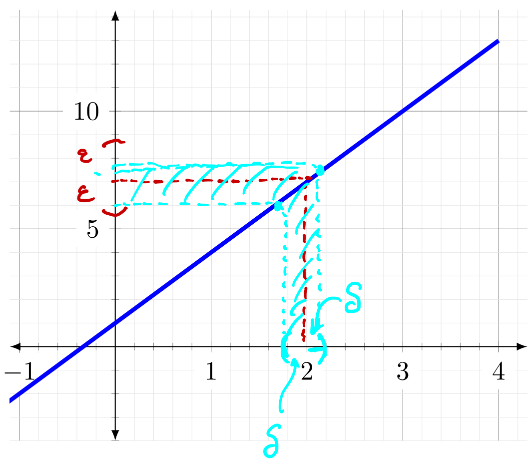
Lecture Notes 2-4: The Precise Definition of the Limit (Day 2)

Review: The Precise Definition of the Limit:

We say $\lim_{x \to a} f(x) = L$, if for *every* number $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

Recall that we used this definition to show that $\lim_{x \to 2} 3x + 1 = 7$.



Algebraically: Given $\varepsilon > 0$, we picked $\delta = \varepsilon/3$.

If $|x-2| < \frac{\varepsilon}{3}$, then

$$\left(-\frac{\varepsilon}{3} < x-2 < \frac{\varepsilon}{3}\right)\cdot 3$$

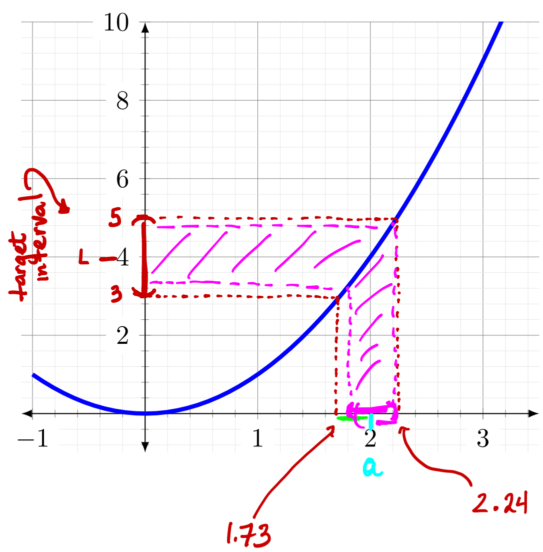
$$-\varepsilon < 3x-6 < \varepsilon \qquad 3x-6 = 3x+1-7 = y-7$$

$$-\varepsilon < y-7 < \varepsilon$$

$|y-7| < \varepsilon$, which is what we wanted to show.

Practice Problems:

1. Let $f(x) = x^2$, graphed below.



(a) Find a number δ such that if $|x - 2| < \delta$, then $|x^2 - 4| < 1$. Use the graph to show that your answer is correct.

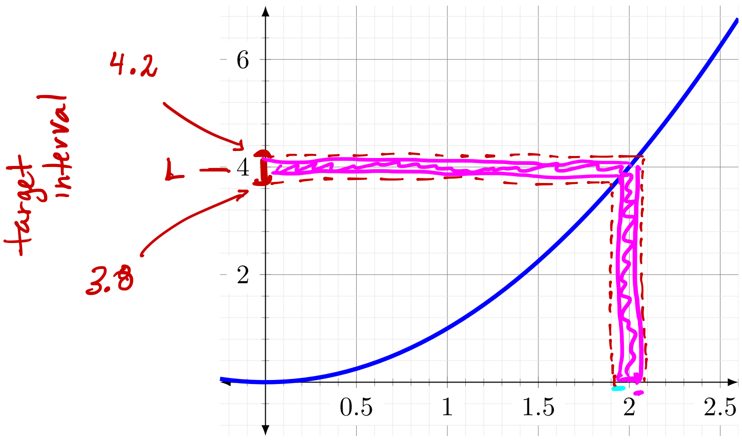
Identify my target: $-1 < x^2 - 4 < 1 \iff 3 < x^2 < 5$

Find endpoints on x-axis: $y = 3, x = \sqrt{3} \approx 1.73$; $y = 5, x = \sqrt{5} \approx 2.24$

$2 - x = 0.27$ $\qquad$ $x - 2 = .24$ ← smaller

Pick $\delta = 0.24$.

Note, any δ so that $0 < \delta \leq 0.24$ will work!



(b) Find a number δ such that if $|x-2| < \delta$, then $|x^2-4| < \frac{1}{5}$. Use the graph to show that your answer is correct.

when $y=4.2$, $x=\sqrt{4.2} \approx 2.0494$, $0.0494 \approx 2-\sqrt{4.2}$

when $y=3.8$, $x=\sqrt{3.8} \approx 1.949$, $0.051 \approx 2-\sqrt{3.8}$

Pick $\delta = 0.0494$

(c) How is finding δ different if the function, $f(x)$, is not linear?

The interval around a isn't symmetric. To find δ, you have to find which side is smaller.

2. A machinist is required to manufacture a metal cube with a volume of $8000\, cm^3$.

(a) What side length produces such a cube? $\boxed{20 \text{ cm}}$

(b) If the machinist is allowed an error tolerance of $\pm 10\, cm^3$ in the volume of the cube, how close to the ideal side length in part (a) must the machinist control the radius?

Our volume must be between $7990\, cm^3$ and $8010\, cm^3$.

So the side length must be between: 19.9917 cm and 20.0083 cm.

The difference of each from 20 is: 0.0083 cm and 0.0083 cm.

Answer: The machinist would need to be within 0.0083 of a centimeter of the ideal length, 20 cm.

(c) In terms of the ϵ-δ definition of $\lim_{x\to a} f(x) = L$, what is:

i. x? length

ii. $f(x)$? Volume

iii. a? 20 cm

iv. L? $8000\, cm^3$

v. ϵ? $10\, cm^3$

vi. δ? 0.0083 cm